Coalescence of geodesics and the BKS midpoint problem in planar first-passage percolation

Thursday, 29 September 2022 21:02

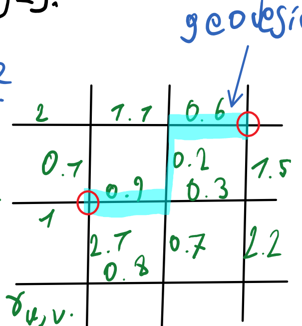
IAS probability seminar
University of Chicago prob. seminar (1 hour)

First-passage percolation (joint with Barbara Dembin and Dor Elboim)

Idea: Random perturbation of Euclidean geometry formed by a random media with short-range correlations.

Model on $\mathbb{Z}^2$: IID $(\tau_e)_{e\in E(\mathbb{Z}^2)}$, $\tau_e \geq 0$ (and τ_e non-deterministic). In this talk assume (partly for simplicity) that their common dist. is abs. cont. and has compact support in $(0,\infty)$. E.g., $\tau_e \sim \mathrm{Unif}[1,2]$.

Passage time: $T_{u,v} := \min\limits_{\substack{p \text{ connects} \\ u \text{ to } v}} \sum_{e\in p} \tau_e$, $u,v\in\mathbb{Z}^2$.



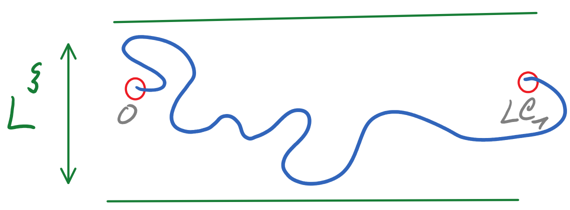
The passage times define a random metric on $\mathbb{Z}^2$.

Geodesic: A path p realizing $T_{u,v}$, denoted $\gamma_{u,v}$.
Existence and uniqueness by abs. cont. assumption.

Goal: Understand the large-scale properties of the metric T. In particular, understand long geodesics.

Basic predictions: Let $v\in\mathbb{Z}^2$. Consider $T_{0,Lv}$ and $\gamma_{0,Lv}$.

$\mathbb{E}(T_{0,Lv}) = \mu(v)L - cL^{\chi}(1+o(1))$,
$\mathrm{Std}(T_{0,Lv}) = c'L^{\chi}(1+o(1))$,
Transversal fluc. of $\gamma_{0,Lv}$ are on scale L^{ξ},



where $\chi = \frac{1}{3}$ and $\xi = \frac{2}{3}$.

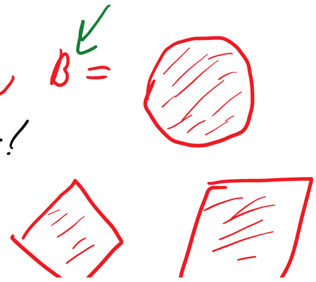
Here $\mu(v)$ is a deterministic norm, $\mu(v) = \lim\limits_{L\to\infty} \frac{T_{0,Lv}}{L}$ a.s. (extended by homogeneity to $\mathbb{R}^2$)
Unit ball $B = \{x\in\mathbb{R}^2 : \mu(x)\leq 1\}$ strictly convex, second derivative of bdry curve always in $(0,\infty)$. Specific shape depends on edge weight dist, possibly never a Euclidean ball

KPZ universality class.

termed limit shape

Known: Norm $\mu(v)$ is well defined, but we do not even know to show that it is never a dilation of the ℓ_1 or ℓ_∞ metric!

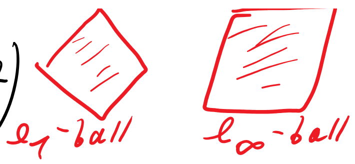
E.g., $B =$



$\mathrm{Var}(T_{0,Lv}) \geq c \log L$ (Newman-Piza 95)
$|T \quad | \leq C \frac{L}{\quad}$ (Benjamini-Kalai-Schramm 02)

$Var(T_{0,Lv}) \geq c \log L$ (Newman-Piza 95)

$Var(T_{0,Lv}) \leq C \frac{L}{\log L}$ (Benjamini-Kalai-Schramm 02, improves $Var(T_{0,Lv}) \leq L$ by Kesten 93 ← Efron-Stein streamlines this)



Version of $\xi \geq \frac{1}{3}$ (Licea-Newman-Piza 96)

Talagrand (or Hoeffding) inequality: $\mathbb{P}(|T_{0,Lv} - \mathbb{E}T_{0,Lv}| \geq t\sqrt{L}) \leq Ce^{-ct^2}$, $\|v\|=1$

Refer to book of Auffinger-Damron-Hanson 15 for more known facts but most questions remain open!

Good understanding only for a related model: directed last-passage perc. where some edge-weight dist. lead to exactly-solved models.

Disordered systems perspective:

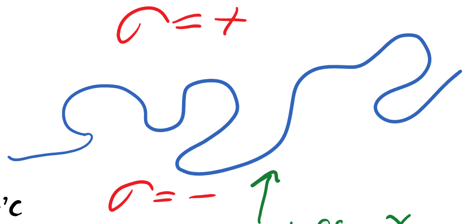
Given non-negative edge weights $\tau = (\tau_e)_{e \in E(\mathbb{Z}^d)}$ as before, the disordered Ising ferromagnet is the model on $\sigma: \mathbb{Z}^d \to \{-1,1\}$ whose formal Hamiltonian is $H^\tau(\sigma) = -\sum_{e=\{u,v\} \in E(\mathbb{Z}^d)} \tau_e \sigma_u \sigma_v$.

Ground configurations are σ whose energy cannot be lowered by changing σ in finitely many places.

Thus, $\sigma \equiv +$ and $\sigma \equiv -$ are ground configurations.

Basic question: Are there non-constant ground configurations?

When $d=2$, they exist iff there exist bigeodesics in the metric T formed from τ.

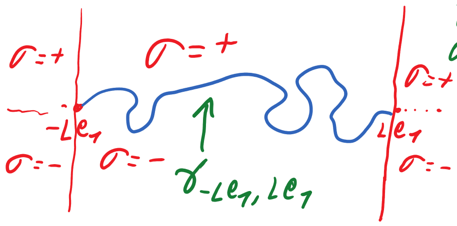


It is predicted that bigeodesics do not exist but this remains open.

One way to try and create a bigeodesic is to use "Dobrushin boundary cond."

$\forall u,v$ along γ, the segment of γ between u,v is the geodesic $\gamma_{u,v}$.

The failure of this is related to the so-called Benjamini-Kalai-Schramm midpoint problem:



Prove that $\forall u,v \in \mathbb{Z}^2$: $\mathbb{P}(\gamma_{u,v}$ passes with dist. 1 of $\frac{u+v}{2}) \to 0$ as $dist(u,v) \to \infty$.

This was solved non-quantitatively by Ahlberg-Hoffman 16 (following Damron-Hanson 15 under diff. limit shape assumption, Alexander 20 under very strong assump.)

Also, a quantitative solution of

Presumably, the prob. decays as $d(u,v)^{-\xi} = d(u,v)^{-2/3}$.

Problem can also be thought of as bounding the influence of specific edge weights on $T_{u,v}$ (this is the BKS perspective).

Our results: (Dembin-Elboim-P. 22)

of specific edge weights on $T_{u,v}$ (this is the BKS perspective).

Our results: (Dembin–Elboim–P. 22)

Limit shape assumption: Let $Sides(B)$ be the number of extreme points of B (infinity if B is not a polygon).
We assume $Sides(B) > 32$.

This seems relatively mild and we can verify that it holds for a class of distributions (perturbations of a constant edge weight)

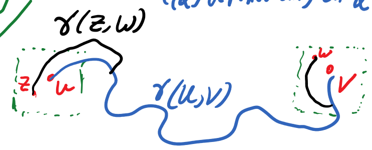
Theorem (coalescence of geodesics):
Let $u, v \in \mathbb{Z}^2$ and write $L = dist(u,v)$. Then $\forall \alpha \in (0, \frac{1}{8})$

(*) $P(\exists z, w$ with $\begin{matrix} d(z,u) \le L^\alpha \\ d(w,v) \le L^\alpha \end{matrix}$ s.t. $|\gamma(z,w) \Delta \gamma(u,v)| > \frac{L}{\log L}) \le C L^{-c(\alpha)}$

symmetric difference

C abs. const., $c(\alpha)$ depends only on α.

have also version with $L^{1-\delta}$



I.e., "coalescence exponent $\ge 1/8$"
presumably, "coalescence exponent $= \xi = 2/3$".
No earlier quantitative work on this exponent except Alexander 20 under very strong assump. verified only in exactly-solved models.

Corollary (BKS midpoint problem):
Let $u, v \in \mathbb{Z}^2$ and write $L = dist(u,v)$. Then

$P(\gamma_{u,v}$ passes within dist. 1 of $\frac{u+v}{2}) \le C L^{-c}$ $\qquad C, c$ absolute constants

Deduction of Corollary: (Assume $\frac{u+v}{2} \in \mathbb{Z}^2$ for notational simplicity)
Let $p := P(\frac{u+v}{2} \in \gamma_{u,v})$. Let δ be the RHS of (*) for $\alpha = 1/16$.
Let $\Lambda = Ball(0, L^{1/16})$ in $\mathbb{Z}^2$. By coalescence, $\forall x \in \Lambda$,

$P(\frac{u+v}{2} \in \gamma_{u+x, v+x}) \ge p - \delta \underset{\text{Translation invariance}}{\Longrightarrow} P(\frac{u+v}{2} - x \in \gamma_{u,v}) \ge p - \delta$.

Thus, for

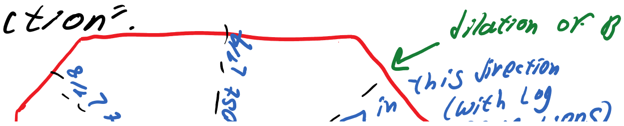
$$\mathbb{E}[|\{x \in \Lambda : \frac{u+v}{2} - x \in \gamma_{u,v}\}|] \ge (p - \delta)|\Lambda|$$

but the quantity in the expectation is always $\le C \cdot \underset{\text{length}}{\text{side}}(\Lambda)$ since the weights are bounded. Thus, $p \le C L^{-c}$.

but cannot spend too much time in green square.

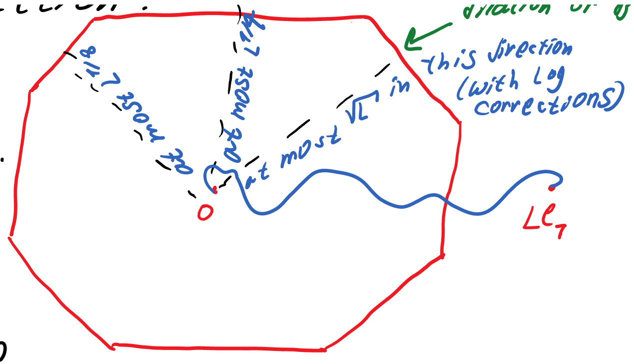
Ingredients in proof of coalescence theorem:

Limit shape sides assumption prevents geodesics from going far in "wrong direction".

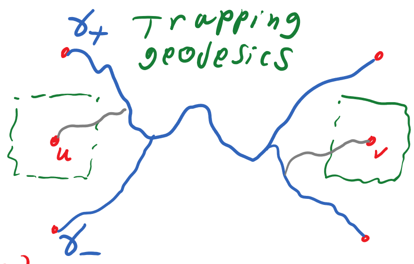


Uses the Gaussian-type extraction of

going far in wrong direction.
Uses the Gaussian-type concentration of the passage time (Talagrand).



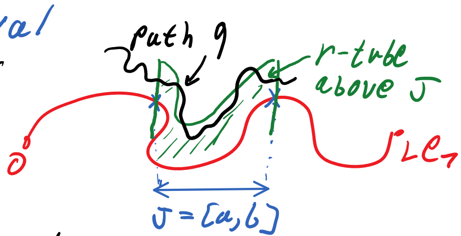
From this we can reduce the coalescence thm. to proving coalescence of specific geodesics that form a "trap" for the others.



Main ingredient (attractive geodesics):

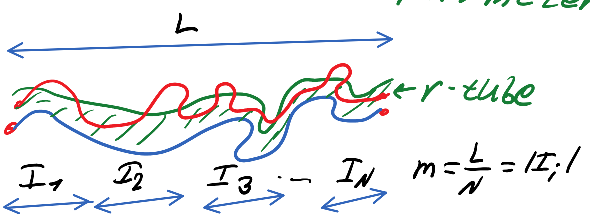
For simplicity, focus on geodesic γ from $(0,0)$ to $(L,0)$.
Let $J=[a,b]\subseteq[0,L]$ satisfy $a,b\in\mathbb{Z}$. Say that a path g in $\mathbb{Z}^2$ is r-close to γ on J if g has a subpath from some (a,y_1) to (b,y_2) in which these endpoints and at least $\frac{1}{2}|J|$ of the edges are at vertical dist. $\le r$ from γ.

Say that J is an attractive interval if all geodesics which are r-close to γ on J intersect γ (above J).



Proposition (attractive geodesics): Let $(I_i)_{i=1}^{N}$ partition $[0,L]$ into intervals of length $m=\frac{L}{N}$.
$\mathbb{P}(\text{More than } \xi N \text{ of the } I_i \text{ are attractive}) \geq 1 - C e^{-c(\log L)^2}$
when, say, $N=m=\sqrt{L}$, $r=L^{1/6}$ and $\xi=\frac{C}{L^{1/12}\log L}$.

Take home message: If two geodesics are close along most of their way then they intersect. Remark that the prop. holds also for first-passage perc. in $\mathbb{Z}^d$, $d\geq 2$ with appropriate parameters



The coalescence result follows from the proposition.
To ensure r-closeness we use the ordering of γ^- and γ^+ and Markov's inequality since ... their average distance.

To ensure ...

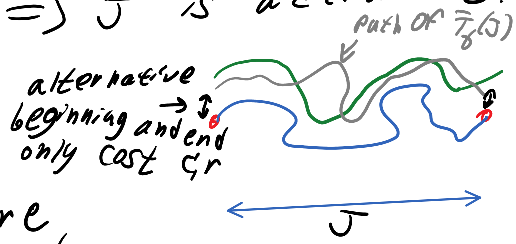
of γ^- and γ^+ and Markov's inequality since we know their average distance.

Planarity

Ideas for attractive geodesics prop. proof:

Let J be an interval. Write $T_\gamma(J)$ for the passage time of γ in its portion above J (from first entrance to first exit).
Write $\overline{T}_\gamma(J)$ for the minimal passage time among paths which are disjoint from γ and r-close to it on J.

Then $\overline{T}_\gamma(J) > T_\gamma(J) + \varsigma r \Rightarrow J$ is attractive.



Ingredients to show this inequality holds for more than ξN of the (I_i):

1) The passage times $T_\gamma(I_i)$ are not much worse than the expected passage times for their endpoints:
Uses Talagrand's concentration inequality and the observation that it "self improves" when considering most (I_i), by concavity of $\sqrt{\cdot}$.

2) For a fixed path p each passage time $\overline{T}_p(I_i)$ is long with non-negligible prob., indep. for separated I_i:
The r-tube of I_i has mr edges. Raising each of their weights by $\approx \frac{1}{\sqrt{mr}}$ doesn't change their dist. much (a Mermin-Wagner style argument).
This raises $\overline{T}_p(I_i)$ by $\sqrt{\frac{m}{r}}$.

3) For each p, conditioning on $\{\gamma = p\}$ only makes the distribution of the weights off p stochastically larger: By Harris' ineq., since $\{\gamma = p\}$ is increasing in these weights.